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APPROXIMATION PROPERTIES OF MULTIVARIATE WAVELETS

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ABSTRACT. Wavelets are generated from refinable functions by using multiresolution analysis. In this paper we investigate the approximation properties of multivariate refinable functions. We give a characterization for the approximation order provided by a refinable function in terms of the order of the sum rules satisfied by the refinement mask. We connect the approximation properties of a refinable function with the spectral properties of the corresponding subdivision and transition operators. Finally, we demonstrate that a refinable function in $W_1^{k-1}(\mathbb{R}^s)$ provides approximation order k.

1. INTRODUCTION

We are concerned with functional equations of the form

$$\phi = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)\phi(M\cdot-\alpha), \tag{1.1}$$

where ϕ is the unknown function defined on the s-dimensional Euclidean space $\mathbb{R}^s$, a is a finitely supported sequence on $\mathbb{Z}^s$, and M is an $s\times s$ integer matrix such that $\lim_{n\to\infty} M^{-n}=0$. The equation (1.1) is called a **refinement equation**, and the matrix M is called a **dilation matrix**. Correspondingly, the sequence a is called the **refinement mask**. Any function satisfying a refinement equation is called a **refinable function**.

If a satisfies

$$\sum_{\alpha\in\mathbb{Z}^s} a(\alpha) = m := |\det M|, \tag{1.2}$$

then it is known that there exists a unique compactly supported distribution ϕ satisfying the refinement equation (1.1) subject to the condition $\hat{\phi}(0)=1$. This distribution is said to be **the normalized solution** to the refinement equation with mask a. This fact was essentially proved by Cavaretta, Dahmen, and Micchelli in [7, Chap. 5] for the case in which the dilation matrix is 2 times the $s\times s$ identity matrix I. The same proof applies to the general refinement equation (1.1).

Wavelets are generated from refinable functions. In [20], Jia and Micchelli discussed how to construct multivariate wavelets from refinable functions associated

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with a general dilation matrix. The approximation and smoothness properties of wavelets are determined by the corresponding refinable functions.

In [9], DeVore, Jawerth, and Popov established a basic theory for nonlinear approximation by wavelets. In their work, the refinement mask was required to be nonnegative. In [15], Jia extended their results and, in particular, removed the restriction of non-negativity of the mask.

Our goal is to characterize the approximation order provided by a refinable function in terms of the refinement mask. This information is important for our understanding of wavelet approximation.

Before proceeding further, we introduce some notation. A multi-index is an s-tuple $\mu = (\mu_1, \dots, \mu_s)$ with its components being nonnegative integers. The length of μ is $|\mu| := \mu_1 + \cdots + \mu_s$, and the factorial of μ is $\mu! := \mu_1! \cdots \mu_s!$. For two multi-indices $\mu = (\mu_1, \dots, \mu_s)$ and $\nu = (\nu_1, \dots, \nu_s)$, we write $\nu \le \mu$ if $\nu_j \le \mu_j$ for $j = 1, \dots, s$. If $\nu \le \mu$, then we define

$$\binom{\mu}{\nu} := \frac{\mu!}{\nu!(\mu-\nu)!}.$$

For $j = 1, \dots, s$, D_j denotes the partial derivative with respect to the jth coordinate. For $\mu = (\mu_1, \dots, \mu_s)$, D^μ is the differential operator $D_1^{\mu_1} \cdots D_s^{\mu_s}$. Moreover, p_μ denotes the monomial given by

$$p_\mu(x) := x_1^{\mu_1} \cdots x_s^{\mu_s}, \qquad x = (x_1, \dots, x_s) \in \mathbb{R}^s.$$

The total degree of p_μ is $|\mu|$. For a nonnegative integer k, we denote by Π_k the linear span of $\{p_\mu : |\mu| \le k\}$. Then $\Pi := \bigcup_{k=0}^\infty \Pi_k$ is the linear space of all polynomials of s variables. We agree that $\Pi_{-1} = \{0\}$.

The Fourier transform of an integrable function f on $\mathbb{R}^s$ is defined by

$$\hat{f}(\xi) = \int_{\mathbb{R}^s} f(x) e^{-ix\cdot\xi}\, dx, \qquad \xi \in \mathbb{R}^s,$$

where $x \cdot \xi$ denotes the inner product of two vectors x and ξ in $\mathbb{R}^s$. The domain of the Fourier transform can be naturally extended to include compactly supported distributions.

We denote by $\ell(\mathbb{Z}^s)$ the linear space of all sequences on $\mathbb{Z}^s$, and by $\ell_0(\mathbb{Z}^s)$ the linear space of all finitely supported sequences on $\mathbb{Z}^s$. For $\alpha \in \mathbb{Z}^s$, we denote by δ_α the element in $\ell_0(\mathbb{Z}^s)$ given by $\delta_\alpha(\alpha) = 1$ and $\delta_\alpha(\beta) = 0$ for all $\beta \in \mathbb{Z}^s \setminus \{\alpha\}$. In particular, we write δ for δ_0. For $j = 1, \dots, s$, let e_j be the jth coordinate unit vector. The difference operator ∇_j on $\ell(\mathbb{Z}^s)$ is defined by $\nabla_j a := a - a(\cdot - e_j)$, $a \in \ell(\mathbb{Z}^s)$. For a multi-index $\mu = (\mu_1, \dots, \mu_s)$, ∇^μ is the difference operator $\nabla_1^{\mu_1} \cdots \nabla_s^{\mu_s}$.

For a compactly supported distribution ϕ on $\mathbb{R}^s$ and a sequence $b \in \ell(\mathbb{Z}^s)$, the **semi-convolution** of ϕ with b is defined by

$$\phi *' b := \sum_{\alpha \in \mathbb{Z}^s} \phi(\cdot - \alpha) b(\alpha).$$

Let $\mathbb{S}(\phi)$ denote the linear space $\{\phi *' b : b \in \ell(\mathbb{Z}^s)\}$. We call $\mathbb{S}(\phi)$ the **shift-invariant space** generated by ϕ. More generally, if Φ is a finite collection of compactly supported distributions on $\mathbb{R}^s$, then we use $\mathbb{S}(\Phi)$ to denote the linear space of all distributions of the form $\sum_{\phi\in\Phi} \phi *' b_\phi$, where $b_\phi \in \ell(\mathbb{Z}^s)$ for $\phi \in \Phi$.

Here is a brief outline of the paper. In Section 2 we clarify the relationship between the order of approximation provided by $\mathbb{S}(\phi)$ and the accuracy of ϕ, the order of the polynomial space contained in $\mathbb{S}(\phi)$. In Section 3 we introduce the so-called sum rules and give a characterization for the accuracy of a refinable function in terms of the order of the sum rules satisfied by the refinement mask. In Section 4, several examples are provided to illustrate the general theory. Section 5 is devoted to a study of the subdivision and transition operators and their applications to approximation properties of refinable functions. Finally, in Section 6, we show that a refinable function in $W_1^k(\mathbb{R}^s)$ associated with an isotropic dilation matrix has accuracy at least $k+1$.

2. Approximation order and polynomial reproducibility

Let ϕ be a compactly supported function in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$. In this section we clarify the relationship between the order of approximation provided by $\mathbb{S}(\phi)$ and the degree of the polynomial space contained in $\mathbb{S}(\phi)$. The reader is referred to [17] for a recent survey on approximation by shift-invariant spaces.

The norm in $L_p(\mathbb{R}^s)$ is denoted by $\|\cdot\|_p$. For an element $f \in L_p(\mathbb{R}^s)$ and a subset G of $L_p(\mathbb{R}^s)$, the distance from f to G, denoted by $\mathrm{dist}_p(f, G)$, is defined by

$$\mathrm{dist}_p(f, G) := \inf_{g \in G} \|f - g\|_p.$$

Let $S := \mathbb{S}(\phi) \cap L_p(\mathbb{R}^s)$. For $h > 0$, let $S^h := \{g(\cdot/h) : g \in S\}$. For a real number $\kappa \ge 0$, we say that $\mathbb{S}(\phi)$ provides **approximation order** κ if for each sufficiently smooth function f in $L_p(\mathbb{R}^s)$, there exists a constant $C > 0$ such that

$$\mathrm{dist}_p(f, S^h) \le C\, h^\kappa \qquad \forall\, h > 0.$$

We say that $\mathbb{S}(\phi)$ provides **density order** κ (see [3]) if for each sufficiently smooth function f in $L_p(\mathbb{R}^s)$,

$$\lim_{h \to 0} \mathrm{dist}_p(f, S^h)/h^\kappa = 0.$$

Let k be a positive integer. Suppose $\mathbb{S}(\phi) \supset \Pi_{k-1}$. Does $\mathbb{S}(\phi)$ always provide approximation order k? The answer is a surprising no. The first counterexample was given by de Boor and Höllig in [4] by considering bivariate C^1-cubics. Their results can be described in terms of box splines.

For a comprehensive study of box splines, the reader is referred to the book [5] by de Boor, Höllig, and Riemenschneider. For our purpose, it suffices to consider the box splines $M_{r,s,t}$ given by

$$\widehat{M}_{r,s,t}(\xi) = \Big(\frac{1 - e^{-i\xi_1}}{i\xi_1}\Big)^r \Big(\frac{1 - e^{-i\xi_2}}{i\xi_2}\Big)^s \Big(\frac{1 - e^{-i(\xi_1+\xi_2)}}{i(\xi_1 + \xi_2)}\Big)^t, \qquad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2,$$

where r, s, and t are nonnegative integers. It is easily seen that $M_{r,s,t} \in L_\infty(\mathbb{R}^2)$ if and only if $\min\{r+s, s+t, t+r\} \ge 1$. Let $\phi_1 := M_{2,1,2}$ and $\phi_2 := M_{1,2,2}$. In [4], de Boor and Höllig proved that $\mathbb{S}(\phi_1, \phi_2) \supseteq \Pi_3$ but $\mathbb{S}(\phi_1, \phi_2)$ does not provide L_∞-approximation order 4. In fact, the optimal L_∞-approximation order provided by $\mathbb{S}(\phi_1, \phi_2)$ is 3. In [21], Ron showed that there exists a compactly supported function ψ in $\mathbb{S}(\phi_1, \phi_2)$ such that $\Pi_3 \subseteq \mathbb{S}(\psi)$. Since $\mathbb{S}(\psi) \subseteq \mathbb{S}(\phi_1, \phi_2)$, the approximation order provided by $\mathbb{S}(\psi)$ is at most 3.

In [6], de Boor and Jia extended the results in [4] in the following way. For $\rho = 1, 2, \ldots$, let k be an integer such that $2\rho + 2 \le k \le 3\rho + 1$. Let

$$\Phi := \{M_{r,s,t} \in C^\rho(\mathbb{R}^2) : r + s + t \le k + 2\}.$$

Then $\mathbb{S}(\Phi) \supset \Pi_k$, but the optimal L_p-approximation order ($1 \le p \le \infty$) provided by $\mathbb{S}(\Phi)$ is k, not $k+1$.

However, if $\mathbb{S}(\phi)$ provides approximation order k, then $\mathbb{S}(\phi)$ contains Π_{k-1}. This was proved by Jia in [16]. Under the additional condition that $\hat{\phi}(0) \ne 0$, it was proved by Ron [21] that $\mathbb{S}(\phi)$ provides L_∞-approximation order k if and only if $\mathbb{S}(\phi)$ contains Π_{k-1}. In general, we have the following results, which were established in [16].

Theorem 2.1. *Let $1 \le p \le \infty$, and let ϕ be a compactly supported function in $L_p(\mathbb{R}^s)$ with $\hat{\phi}(0) \ne 0$. For every positive integer k, the following statements are equivalent:*
(a) $\mathbb{S}(\phi)$ provides approximation order k.
(b) $\mathbb{S}(\phi)$ provides density order $k-1$.
(c) $\mathbb{S}(\phi)$ contains Π_{k-1}.
(d) $D^\mu \hat{\phi}(2\pi\beta) = 0$ for all μ with $|\mu| \le k-1$ and all $\beta \in \mathbb{Z}^s \setminus \{0\}$.

We remark that the implications (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (d) are valid without the assumption $\hat{\phi}(0) \ne 0$. Indeed, (a) $\Rightarrow$ (b) is obvious, (b) $\Rightarrow$ (c) was proved in [16], and the implication (c) $\Rightarrow$ (d) was established in [2].

Suppose ϕ is the normalized solution of the refinement equation (1.1). If ϕ lies in $L_p(\mathbb{R}^s)$ for some p, $1 \le p \le \infty$, then Theorem 2.1 applies to ϕ, because $\hat{\phi}(0) = 1$. Thus, there are two questions of interest. The first question is how to determine whether ϕ lies in $L_p(\mathbb{R}^s)$, and the second problem is how to characterize the highest degree of polynomials contained in $\mathbb{S}(\phi)$. The first question was discussed by Han and Jia in [12]. In this paper, we concentrate on the second question. When we speak of polynomial containment, ϕ is not required to be an integrable function. Thus, we say that a compactly supported distribution ϕ on $\mathbb{R}^s$ has **accuracy** k, if $\mathbb{S}(\phi) \supset \Pi_{k-1}$ (see [13] for the terminology of accuracy).

We point out that the equivalence between (c) and (d) in Theorem 2.1 remains true for every compactly supported distribution ϕ on $\mathbb{R}^s$.

If ϕ is a compactly supported continuous function on $\mathbb{R}^s$, and if ϕ satisfies condition (d), then it was proved in [14] that

$$\phi *' p = \hat{\phi}(-iD)\, p \qquad \forall\, p \in \Pi_{k-1}, \tag{2.1}$$

where i is the imaginary unit and $\hat{\phi}(-iD)$ denotes the differential operator given by the formal power series

$$\sum_{\mu \ge 0} \frac{D^\mu \hat{\phi}(0)}{\mu!}(-iD)^\mu.$$

For a given polynomial p, $D^\mu p = 0$ if $|\mu|$ is sufficiently large. Thus, $\hat{\phi}(-iD)$ is well defined on Π. We indicate that (2.1) is also valid for a compactly supported distribution ϕ on $\mathbb{R}^s$ satisfying condition (d). To see this, choose a function $\rho \in C_c^\infty(\mathbb{R}^s)$ such that $\hat{\rho}(0) = 1$ and $D^\nu \hat{\rho}(0) = 0$ for all ν with $0 < |\nu| \le k-1$. Let $\rho_n := \rho(\cdot/n)/n^s$ for $n = 1, 2, \ldots$. Then for each n, $\phi_n := \phi * \rho_n$, the convolution of ϕ with ρ_n, is a function in $C_c^\infty(\mathbb{R}^s)$. Moreover, the sequence $(\phi_n)_{n=1,2,\ldots}$ converges to ϕ in the sense that

$$\lim_{n\to\infty} \langle \phi_n, f\rangle = \langle \phi, f\rangle \qquad \forall\, f \in C_c^\infty(\mathbb{R}^s).$$

See [1, p. 97] for these facts. Thus, we have $\hat{\phi}_n(\xi) = \hat{\phi}(\xi)\hat{\rho}_n(\xi)$ for $\xi \in \mathbb{R}^s$. Since ϕ satisfies condition (d), by using the Leibniz formula for differentiation, we get $D^\mu \hat{\phi}_n(2\pi\beta) = 0$ for $|\mu| \le k-1$ and $\beta \in \mathbb{Z}^s \setminus \{0\}$. Hence (2.1) is applicable to ϕ_n and

$$\phi_n *' p = \hat{\phi}_n(-iD)\, p \qquad \forall p \in \Pi_{k-1}.$$

Letting $n \to \infty$ in the above equation, we obtain $\phi *' p = \hat{\phi}(-iD)\, p$ for all $p \in \Pi_{k-1}$. Consequently, the linear mapping $\phi *'$ given by $p \mapsto \phi *' p$ maps Π_{k-1} to Π_{k-1}. If, in addition, $\hat{\phi}(0) \ne 0$, then this mapping is one-to-one, and hence it is onto. This shows that (d) $\Rightarrow$ (c) is valid for every compactly supported distribution ϕ on $\mathbb{R}^s$ with $\hat{\phi}(0) \ne 0$.

Next, we show that (c) $\Rightarrow$ (d) for every compactly supported distribution ϕ on $\mathbb{R}^s$. If ϕ is a compactly supported continuous function on $\mathbb{R}^s$, this was proved in [2] and [14]. Let ϕ be a compactly supported distribution on $\mathbb{R}^s$. For a fixed element $\beta \in \mathbb{Z}^s \setminus \{0\}$, choose a function $\rho \in C_c^\infty(\mathbb{R}^s)$ such that $\hat{\rho}(0) \ne 0$ and $\hat{\rho}(2\pi\beta) \ne 0$. Then the convolution $\phi * \rho$ is a function in $C_c^\infty(\mathbb{R}^s)$ and its Fourier transform is $\hat{\phi}\hat{\rho}$. Note that the mapping $\rho *$ given by $q \mapsto \rho * q$ maps Π_{k-1} to Π_{k-1}. Since $\hat{\rho}(0) \ne 0$, this mapping is one-to-one; hence it is onto. Thus, for $p \in \Pi_{k-1}$, we can find $q \in \Pi_{k-1}$ such that $p = \rho * q$. Since $\mathbb{S}(\phi) \supset \Pi_{k-1}$, there exists some $b \in \ell(\mathbb{Z}^s)$ such that $q = \phi *' b$. It follows that $p = \rho * (\phi *' b) = (\rho * \phi) *' b$. This shows that $\mathbb{S}(\phi * \rho) \supset \Pi_{k-1}$. By what has been proved, $D^\mu(\hat{\phi}\hat{\rho})(2\pi\beta) = 0$ for all μ with $|\mu| \le k-1$. Since $\hat{\rho}(2\pi\beta) \ne 0$, we can write $\hat{\phi} = (\hat{\phi}\hat{\rho})(1/\hat{\rho})$ in a neighborhood of $2\pi\beta$. By applying the Leibniz formula for differentiation to this equation, we obtain $D^\mu \hat{\phi}(2\pi\beta) = 0$ for $|\mu| \le k-1$. This shows that (c) $\Rightarrow$ (d) for every compactly supported distribution ϕ on $\mathbb{R}^s$.

To summarize, a compactly supported distribution ϕ on $\mathbb{R}^s$ with $\hat{\phi}(0) \ne 0$ possesses accuracy k if and only if $D^\mu \hat{\phi}(2\pi\beta) = 0$ for all μ with $|\mu| \le k-1$ and all $\beta \in \mathbb{Z}^s \setminus \{0\}$.

3. Characterization of accuracy

The purpose of this section is to give a characterization for the accuracy of a refinable function in terms of the refinement mask.

For an $s \times s$ dilation matrix M, let Γ be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s / M\mathbb{Z}^s$, and let Ω be a complete set of representatives of the distinct cosets of $\mathbb{Z}^s / M^T\mathbb{Z}^s$, where M^T denotes the transpose of M. Evidently, $\#\Gamma = \#\Omega = |\det M|$. Without loss of any generality, we may assume that $0 \in \Gamma$ and $0 \in \Omega$.

Suppose a is a finitely supported sequence on $\mathbb{Z}^s$ satisfying (1.2). Let ϕ be the normalized solution of the refinement equation (1.1). Taking Fourier transform of both sides of (1.1), we obtain

$$\hat{\phi}(\xi) = H((M^T)^{-1}\xi)\, \hat{\phi}((M^T)^{-1}\xi), \qquad \xi \in \mathbb{R}^s, \tag{3.1}$$

where

$$H(\xi) := \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) e^{-i\alpha\cdot\xi}/m, \qquad \xi \in \mathbb{R}^s. \tag{3.2}$$

Note that H is a 2π-periodic function and $H(0) = 1$.

For a compactly supported distribution ϕ on $\mathbb{R}^s$, define

$$N(\phi) := \{\xi \in \mathbb{R}^s : \hat{\phi}(\xi + 2\pi\beta) = 0 \ \forall \beta \in \mathbb{Z}^s\}.$$

If ϕ is a compactly supported function in $L_p(\mathbb{R}^s)$ $(1 \le p \le \infty)$, then the shifts of ϕ are stable if and only if $N(\phi)$ is the empty set (see [19]).

Theorem 3.1. *Let a be a finitely supported sequence on $\mathbb{Z}^s$ satisfying (1.2), and let H be the function given in (3.2). If*

$$D^\mu H\big(2\pi(M^T)^{-1}\omega\big) = 0 \qquad \forall \omega \in \Omega \setminus \{0\} \text{ and } |\mu| \le k-1, \tag{3.3}$$

then the normalized solution ϕ of the refinement equation (1.1) has accuracy k. Conversely, if ϕ has accuracy k, and if $N(\phi) \cap (2\pi(M^T)^{-1}\Omega) = \emptyset$, then (3.3) holds true.

Proof. Suppose that (3.3) is satisfied. Since H is 2π-periodic, (3.3) implies

$$D^\mu H\big(2\pi(M^T)^{-1}\beta\big) = 0 \qquad \forall \beta \in \mathbb{Z}^s \setminus (M^T\mathbb{Z}^s) \text{ and } |\mu| \le k-1. \tag{3.4}$$

Let f and g be the functions given by

$$f(\xi) := H\big((M^T)^{-1}\xi\big) \quad \text{and} \quad g(\xi) := \hat{\phi}\big((M^T)^{-1}\xi\big), \qquad \xi \in \mathbb{R}^s.$$

For $|\mu| \le k-1$ and $\beta \in \mathbb{Z}^s \setminus \{0\}$, applying the Leibniz formula for differentiation to (3.1), we obtain

$$D^\mu \hat{\phi}(2\pi\beta) = \sum_{\nu \le \mu} \binom{\mu}{\nu} D^\nu f(2\pi\beta)\, D^{\mu-\nu} g(2\pi\beta). \tag{3.5}$$

By using the chain rule, we see that $D^\nu f(2\pi\beta)$ is a linear combination of terms of the form $D^\alpha H(2\pi(M^T)^{-1}\beta)$, where $\alpha \le \nu$. In light of (3.4), these terms are equal to 0 if $\beta \in \mathbb{Z}^s \setminus (M^T\mathbb{Z}^s)$. This shows that $D^\mu\hat{\phi}(2\pi\beta) = 0$ for $\beta \in \mathbb{Z}^s \setminus (M^T\mathbb{Z}^s)$.

We shall prove that, for $r = 0, 1, \ldots,$ $D^\mu\hat{\phi}(2\pi\beta) = 0$ for $\beta \in ((M^T)^r\mathbb{Z}^s) \setminus ((M^T)^{r+1}\mathbb{Z}^s)$. This will be done by induction on r. The case $r = 0$ was established above. Suppose $r \ge 1$ and our claim has been verified for $r-1$. Let $\beta \in ((M^T)^r\mathbb{Z}^s) \setminus ((M^T)^{r+1}\mathbb{Z}^s)$. Then we have $(M^T)^{-1}\beta \in ((M^T)^{r-1}\mathbb{Z}^s) \setminus ((M^T)^r\mathbb{Z}^s)$. Hence, by the induction hypothesis, $D^\mu\hat{\phi}(2\pi(M^T)^{-1}\beta) = 0$ for $|\mu| \le k-1$. Consequently, $D^\mu g(2\pi\beta) = 0$ for all μ with $|\mu| \le k-1$. This in connection with (3.5) tells us that $D^\mu\hat{\phi}(2\pi\beta) = 0$ for $|\mu| \le k-1$, thereby completing the induction procedure. The sufficiency part of the theorem has been established.

Conversely, suppose ϕ has accuracy k and $N(\phi) \cap (2\pi(M^T)^{-1}\Omega) = \emptyset$. Then

$$D^\mu\hat{\phi}(2\pi\beta) = 0 \qquad \forall \beta \in \mathbb{Z}^s \setminus \{0\} \text{ and } |\mu| \le k-1.$$

Let $\omega \in \Omega \setminus \{0\}$. Since $N(\phi) \cap (2\pi(M^T)^{-1}\Omega) = \emptyset$, there exists some $\beta \in \mathbb{Z}^s$ such that $\hat{\phi}(\gamma) \ne 0$ for $\gamma := 2\pi\beta + 2\pi(M^T)^{-1}\omega$. Thus, the following identity is valid for ξ in a neighborhood of γ:

$$H(\xi) = \hat{\phi}(M^T\xi)\big[1/\hat{\phi}(\xi)\big].$$

Let h be the function given by $\xi \mapsto \hat{\phi}(M^T\xi)$, $\xi \in \mathbb{R}^s$. By using the Leibniz formula for differentiation, we obtain

$$D^\mu H(\gamma) = \sum_{\nu \le \mu} \binom{\mu}{\nu} D^\nu h(\gamma)\, D^{\mu-\nu}\big[1/\hat{\phi}\big](\gamma).$$

By the chain rule, $D^\nu h(\gamma)$ is a linear combination of terms of the form $D^\alpha \hat{\phi}(M^T\gamma)$, where $\alpha \le \nu$. Note that

$$M^T\gamma = M^T(2\pi\beta + 2\pi(M^T)^{-1}\omega) = 2\pi(M^T)\,\beta + 2\pi\omega \in 2\pi\mathbb{Z}^s \setminus \{0\}.$$

Hence $D^\alpha\hat{\phi}(M^T\gamma) = 0$ for $|\alpha| \le k-1$, because ϕ has accuracy k. Therefore we obtain $D^\mu H(2\pi\beta + 2\pi(M^T)^{-1}\omega) = 0$ for $|\mu| \le k-1$. But H is 2π-periodic. This shows that $D^\mu H(2\pi(M^T)^{-1}\omega) = 0$ for all $\omega \in \Omega \setminus \{0\}$ and $|\mu| \le k-1$, as desired. The proof of the theorem is complete. □

In the rest of this section we shall show that (3.3) is equivalent to saying that, for all $p \in \Pi_{k-1}$,

$$\sum_{\beta\in\mathbb{Z}^s} a(M\beta)\,p(M\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)\,p(M\beta+\gamma) \qquad \forall\,\gamma\in\Gamma. \tag{3.6}$$

For this purpose, we first establish the following lemma.

Lemma 3.2. *The matrix*

$$\frac{1}{\sqrt{m}}\big(e^{i2\pi M^{-1}\gamma\cdot\omega}\big)_{\gamma\in\Gamma,\omega\in\Omega} \tag{3.7}$$

is a unitary one.

Proof. Let $\gamma \in \Gamma\setminus\{0\}$. We claim that there exists some $\omega' \in \Omega$ such that $M^{-1}\gamma\cdot\omega' \notin \mathbb{Z}$. Any element $\beta \in \mathbb{Z}^s$ can be represented as $M^T\alpha + \omega$ for some $\alpha \in \mathbb{Z}^s$ and $\omega \in \Omega$. Note that $(M^{-1}\gamma)\cdot(M^T\alpha) = \gamma\cdot\alpha \in \mathbb{Z}$ for all $\alpha \in \mathbb{Z}^s$. Hence $M^{-1}\gamma\cdot\omega' \in \mathbb{Z}$ for all $\omega' \in \Omega$ implies that $M^{-1}\gamma\cdot\beta \in \mathbb{Z}$ for all $\beta \in \mathbb{Z}^s$. In other words, $M^{-1}\gamma \in \mathbb{Z}^s$, and hence $\gamma \in M\mathbb{Z}^s$, which contradicts the assumption $\gamma \in \Gamma\setminus\{0\}$. This verifies our claim.

For a fixed element γ in $\Gamma\setminus\{0\}$, let

$$\sigma := \sum_{\omega\in\Omega} e^{i2\pi M^{-1}\gamma\cdot\omega}.$$

Choose $\omega' \in \Omega$ such that $M^{-1}\gamma\cdot\omega' \notin \mathbb{Z}$. We have

$$e^{i2\pi M^{-1}\gamma\cdot\omega'}\sigma = \sum_{\omega\in\Omega} e^{i2\pi(M^{-1}\gamma)\cdot(\omega+\omega')} = \sum_{\omega\in\Omega} e^{i2\pi M^{-1}\gamma\cdot\omega} = \sigma.$$

Since $e^{i2\pi M^{-1}\gamma\cdot\omega'} \ne 1$, it follows that $\sigma = 0$. This shows that

$$\sum_{\omega\in\Omega} e^{i2\pi M^{-1}\gamma\cdot\omega} = 0 \qquad \forall\,\gamma\in\Gamma\setminus\{0\}. \tag{3.8}$$

Similarly, we can prove that

$$\sum_{\gamma\in\Gamma} e^{i2\pi M^{-1}\gamma\cdot\omega} = 0 \qquad \forall\,\omega\in\Omega\setminus\{0\}. \tag{3.9}$$

Finally, the matrix in (3.7) is unitary if and only if for every pair of elements $\gamma, \gamma' \in \Gamma$,

$$\frac{1}{m}\sum_{\omega\in\Omega} e^{i2\pi M^{-1}(\gamma-\gamma')\cdot\omega} = \begin{cases} 1 & \text{if } \gamma = \gamma', \\ 0 & \text{if } \gamma \ne \gamma'. \end{cases}$$

For $\gamma = \gamma'$, this comes from the fact $\#\Omega = m$; for $\gamma \ne \gamma'$, this follows from (3.8). □

Lemma 3.3. *Let a be a finitely supported sequence satisfying (1.2), and let H be the function given in (3.2). Then the following two conditions are equivalent for every polynomial p:*
(a) $p(iD)\, H(2\pi(M^T)^{-1}\omega) = 0$ for all $\omega \in \Omega \setminus \{0\}$.
(b) $\sum_{\beta\in\mathbb{Z}^s} a(M\beta)\, p(M\beta) = \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)\, p(M\beta+\gamma)$ for all $\gamma \in \Gamma$.

Proof. By (3.2) we have

$$m\, p(iD)H(\xi) = \sum_{\alpha\in\mathbb{Z}^s} a(\alpha)p(\alpha)e^{-i\alpha\cdot\xi}, \qquad \xi \in \mathbb{R}^s.$$

An element $\alpha \in \mathbb{Z}^s$ can be written uniquely as $M\beta + \gamma$ with $\beta \in \mathbb{Z}^s$ and $\gamma \in \Gamma$. Observe that, for $\xi := 2\pi(M^T)^{-1}\omega$,

$$-i\alpha\cdot\xi = -i(M\beta+\gamma)\cdot 2\pi(M^T)^{-1}\omega = -i\, 2\pi\beta\cdot\omega - i\, 2\pi\gamma\cdot(M^T)^{-1}\omega.$$

Hence we have

$$m\, p(iD)H(2\pi(M^T)^{-1}\omega) = \sum_{\gamma\in\Gamma} b(\gamma)e^{-i2\pi\gamma\cdot(M^T)^{-1}\omega}, \tag{3.10}$$

where

$$b(\gamma) := \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)\, p(M\beta+\gamma).$$

Condition (b) says that $b(\gamma) = b(0)$ for all $\gamma \in \Gamma$. Hence by (3.9) we deduce from (3.10) that

$$m\, p(iD)H(2\pi(M^T)^{-1}\omega) = b(0)\sum_{\gamma\in\Gamma} e^{-i2\pi\gamma\cdot(M^T)^{-1}\omega} = 0$$

for all $\omega \in \Omega \setminus \{0\}$. This shows that (b) $\Rightarrow$ (a).

Conversely, (3.10) tells us that condition (a) implies

$$\sum_{\gamma\in\Gamma} b(\gamma)e^{-i2\pi M^{-1}\gamma\cdot\omega} = 0 \qquad \forall \omega \in \Omega \setminus \{0\}.$$

Let η be an element of Γ. Then it follows that

$$\sum_{\omega\in\Omega} e^{i2\pi M^{-1}\eta\cdot\omega} \sum_{\gamma\in\Gamma} b(\gamma)e^{-i2\pi M^{-1}\gamma\cdot\omega} = \sum_{\gamma\in\Gamma} b(\gamma).$$

On the other hand,

$$\sum_{\omega\in\Omega} e^{i2\pi M^{-1}\eta\cdot\omega} \sum_{\gamma\in\Gamma} b(\gamma)e^{-i2\pi M^{-1}\gamma\cdot\omega} = \sum_{\gamma\in\Gamma} b(\gamma) \sum_{\omega\in\Omega} e^{i2\pi M^{-1}(\eta-\gamma)\cdot\omega} = m\, b(\eta),$$

since $\sum_{\omega\in\Omega} e^{i2\pi M^{-1}(\eta-\gamma)\cdot\omega} = 0$ for $\gamma \neq \eta$, by Lemma 3.2. This shows $m\, b(\eta) = \sum_{\gamma\in\Gamma} b(\gamma)$. Therefore $b(\eta) = b(0)$ for all $\eta \in \Gamma$. In other words, (a) implies (b). $\square$

If an element $a \in \ell_0(\mathbb{Z}^s)$ satisfies (3.6) for all $p \in \Pi_{k-1}$, then we say that a satisfies the **sum rules** of order k. The results of this section can be summarized as follows: If the refinement mask a satisfies the sum rules of order k, then the normalized solution ϕ of the refinement equation with mask a has accuracy k. Conversely, if ϕ has accuracy k, and if $N(\phi) \cap (2\pi(M^T)^{-1}\Omega) = \emptyset$, then a satisfies the sum rules of order k.

4. Examples

In this section we give several examples to illustrate the general theory.

The **symbol** of a sequence $a \in \ell_0(\mathbb{Z}^s)$ is the Laurent polynomial $\tilde{a}(z)$ given by

$$\tilde{a}(z) := \sum_{\alpha \in \mathbb{Z}^s} a(\alpha) z^\alpha, \qquad z \in (\mathbb{C} \setminus \{0\})^s,$$

where $z^\alpha := z_1^{\alpha_1} \cdots z_s^{\alpha_s}$ for $z = (z_1, \ldots, z_s) \in \mathbb{C}^s$ and $\alpha = (\alpha_1, \ldots, \alpha_s) \in \mathbb{Z}^s$. If a is supported on $[0, N]^s$ for some positive integer N, then $\tilde{a}(z)$ is a polynomial of z.

In the univariate case ($s = 1$), if a satisfies the sum rules of order k, then $\tilde{a}(z)$ is divisible by $(1+z)^k$ (see, e.g., [8]). In the multivariate case ($s > 1$), this is no longer true.

Example 4.1. Let $s = 2$ and $M = 2I$, where I is the 2×2 identity matrix. Let a be the sequence on $\mathbb{Z}^2$ given by its symbol

$$\tilde{a}(z) := z_1^2 + z_2 + z_1 z_2 + z_1 z_2^2.$$

Then a satisfies the sum rules of order 1. But the polynomial $\tilde{a}(z)$ is irreducible.

It is easy to verify that a satisfies the sum rules of order 1. Let us show that $\tilde{a}(z)$ is irreducible. Suppose to the contrary that $\tilde{a}(z)$ is reducible. Then $\tilde{a}(z)$ can be factored as

$$\tilde{a}(z) = f(z) g(z),$$

where f and g are polynomials of (total) degree at least 1. Since the degree of $\tilde{a}(z)$ is 3, the degree of either f or g is 1. Suppose the degree of f is 1 and

$$f(z_1, z_2) = \lambda z_1 + \mu z_2 + \nu,$$

where λ, μ, ν are complex numbers and either $\lambda \neq 0$ or $\mu \neq 0$. If $\lambda \neq 0$, then for all $z_2 \in \mathbb{C}$, $f(-(\mu z_2 + \nu)/\lambda, z_2) = 0$, and so

$$\tilde{a}\big(-(\mu z_2 + \nu)/\lambda, z_2\big) = 0 \qquad \forall\, z_2 \in \mathbb{C}.$$

If $\mu \neq 0$, then $\tilde{a}(-(\mu z_2 + \nu)/\lambda, z_2)$ is a polynomial of z_2 of degree 3 with $-\mu/\lambda$ being its leading coefficient. Hence $\mu = 0$. But it is also impossible that $\tilde{a}(-\nu/\lambda, z_2) = 0$ for all $z_2 \in \mathbb{C}$. Thus, we must have $\lambda = 0$, and hence $\tilde{a}(z_1, -\nu/\mu) = 0$ for all $z_1 \in \mathbb{C}$. However, $\tilde{a}(z_1, -\nu/\mu)$ is a polynomial of z_1 of degree 2 with 1 being its leading coefficient. This contradiction shows that $\tilde{a}(z)$ is irreducible.

Let a be the sequence given as above, and let ϕ be the normalized solution of the refinement equation

$$\phi = \sum_{\alpha \in \mathbb{Z}^2} a(\alpha) \phi(2 \cdot - \alpha).$$

Then ϕ lies in $L_2(\mathbb{R}^2)$. This can be verified by using the results in [12]. Let b be the element in $\ell_0(\mathbb{Z}^2)$ given by its symbol

$$\tilde{b}(z) := |\tilde{a}(z)|^2/4 \qquad \text{for } |z_1| = 1 \text{ and } |z_2| = 1.$$

We have

$$\begin{aligned} 4\,\tilde{b}(z) = 4 &+ z_1 + z_1^{-1} + z_2 + z_2^{-1} + z_1 z_2 + z_1^{-1} z_2^{-1} \\ &+ z_1 z_2^{-1} + z_1^{-1} z_2 + z_1 z_2^{-2} + z_1^{-1} z_2^2 + z_1^2 z_2^{-1} + z_1^{-2} z_2. \end{aligned}$$

Let B be the linear operator on $\ell_0(\mathbb{Z}^2)$ given by

$$Bv(\alpha) := \sum_{\beta\in\mathbb{Z}^2} b(2\alpha-\beta)\, v(\beta), \qquad \alpha \in \mathbb{Z}^2,$$

where $v \in \ell_0(\mathbb{Z}^2)$. Let W be the B-invariant subspace generated by $-\delta_{-e_1}+2\delta-\delta_{e_1}$ and $-\delta_{-e_2}+2\delta-\delta_{e_2}$. Then the spectral radius ρ of the linear operator $B|_W$ is $3/4$. Since $\rho<1$, by [12, Theorems 3.3 and 4.1], the subdivision scheme associated with a is L_2-convergent. Therefore, $\phi \in L_2(\mathbb{R}^2)$ and the shifts of ϕ are orthonormal (see [11]). We conclude that the optimal order of approximation provided by $\mathbb{S}(\phi)$ is 1.

If the refinement mask a satisfies the sum rules of order k, then the normalized solution ϕ of the refinement equation with mask a has accuracy k. However, if the condition $N(\phi)\cap(2\pi(M^T)^{-1}\Omega)=\emptyset$ is not satisfied, then ϕ could have higher accuracy. For instance, the function ϕ on $\mathbb{R}$ given by $\phi(x)=1/2$ for $0\le x<2$ and $\phi(x)=0$ for $x\in\mathbb{R}\setminus[0,2)$ satisfies the refinement equation

$$\phi = \sum_{\alpha\in\mathbb{Z}} a(\alpha)\phi(2\cdot-\alpha),$$

where the symbol of the mask a is $\tilde{a}(z)=1+z^2$. Then a does not satisfy the sum rules of order 1. But ϕ has accuracy 1, and $\mathbb{S}(\phi)$ provides L_∞-approximation order 1. The following is an example in the two-dimensional case.

Example 4.2. Let ϕ be the Zwart-Powell element defined by its Fourier transform

$$\hat{\phi}(\xi_1,\xi_2) := g(\xi_1)\, g(\xi_2)\, g(\xi_1+\xi_2)\, g(-\xi_1+\xi_2), \qquad (\xi_1,\xi_2)\in\mathbb{R}^2,$$

where g is the function on $\mathbb{R}$ given by $\xi\mapsto(1-e^{-i\xi})/(i\xi)$, $\xi\in\mathbb{R}$. Then ϕ is a compactly supported continuous function on $\mathbb{R}^2$ and $\mathbb{S}(\phi)$ provides L_∞-approximation order 3. On the other hand, ϕ is refinable but the corresponding mask does not satisfy the sum rules of order 3.

For the first statement the reader is referred to [5, p. 72]. Let us verify the second statement. From [5, p. 140] we know that the Zwart-Powell element ϕ is refinable and the corresponding mask a is given by $a(\alpha)=0$ for $\alpha\in\mathbb{Z}^2\setminus[-1,2]\times[0,3]$ and

$$\big(a(\alpha_1,\alpha_2)\big)_{-1\le\alpha_1\le2,0\le\alpha_2\le3} = \frac{1}{4}\begin{bmatrix} 0 & 1 & 1 & 0\\ 1 & 2 & 2 & 1\\ 1 & 2 & 2 & 1\\ 0 & 1 & 1 & 0\end{bmatrix}.$$

Evidently, the mask a satisfies the sum rules of order 2, but a does not satisfy the sum rules of order 3. Note that $(\pi,\pi)\in N(\phi)$ in this case.

Example 4.3. Let M be the matrix

$$\begin{pmatrix} 1 & -1\\ 1 & 1\end{pmatrix},$$

and let a be the sequence on $\mathbb{Z}^2$ such that $a(\alpha)=0$ for $\alpha\in\mathbb{Z}^2\setminus[-2,2]^2$ and

$$\big(a(\alpha_1,\alpha_2)\big)_{-2\le\alpha_1,\alpha_2\le2} = \frac{1}{32}\begin{bmatrix} 0 & -1 & 0 & -1 & 0\\ -1 & 0 & 10 & 0 & -1\\ 0 & 10 & 32 & 10 & 0\\ -1 & 0 & 10 & 0 & -1\\ 0 & -1 & 0 & -1 & 0\end{bmatrix}.$$

Let ϕ be the normalized solution of the refinement equation (1.1) with mask a and dilation matrix M given as above. Then ϕ is a compactly supported continuous function on $\mathbb{R}^2$, and the optimal approximation order provided by $\mathbb{S}(\phi)$ is 4.

Let us verify that a satisfies the sum rules of order 4. We observe that $\alpha = (\alpha_1, \alpha_2)$ lies in $M\mathbb{Z}^2$ if and only if $\alpha_1 + \alpha_2$ is an even integer. Hence the sum rule for a polynomial p of two variables reads as follows:

$$\sum_{\alpha_1+\alpha_2\in 2\mathbb{Z}} p(\alpha)a(\alpha) = \sum_{\beta_1+\beta_2\notin 2\mathbb{Z}} p(\beta)a(\beta),$$

that is,

$$32\,p(0,0) = 10 \sum_{|\alpha_1|+|\alpha_2|=1} p(\alpha_1,\alpha_2) - \sum_{|\alpha_1|+|\alpha_2|=3} p(\alpha_1,\alpha_2).$$

We can easily verify that this condition is satisfied for all $p \in \Pi_3$, but it is not satisfied for the monomial p given by $p(x_1, x_2) = x_1^2 x_2^2$, $(x_1, x_2) \in \mathbb{R}^2$. Therefore the refinement mask a satisfies the sum rules of order 4, but not of order 5.

In the present case, $\Omega := \{(0,0), (1,0)\}$ is a complete set of representatives of the distinct cosets of $\mathbb{Z}^2/M^T\mathbb{Z}^2$. We have $2\pi(M^T)^{-1}\Omega = \{(0,0), (\pi,\pi)\}$. Since $\hat{\phi}(0,0) = 1$, in order to verify the condition $N(\phi) \cap (2\pi(M^T)^{-1}\Omega) = \emptyset$, it suffices to show that $\hat{\phi}(\pi,\pi) \neq 0$. For this purpose, we observe that

$$\hat{\phi}(\xi) = \prod_{k=1}^{\infty} H\big((M^T)^{-k}\xi\big), \qquad \xi \in \mathbb{R}^2,$$

where

$$H(\xi) = \big[32 + 20(\cos\xi_1 + \cos\xi_2) - 4\cos(2\xi_1 + \xi_2) - 4\cos(\xi_1 + 2\xi_2)\big]/64, \qquad \xi = (\xi_1, \xi_2) \in \mathbb{R}^2.$$

We have $(M^T)^{-1}(\pi,\pi)^T = (0,\pi)^T$ and $H(0,\pi) > 0$. Suppose

$$(\eta_1, \eta_2)^T = (M^T)^{-k}(\pi,\pi)^T$$

for some integer $k \geq 2$. Then $|\eta_1| \leq \pi/2$ and $|\eta_2| \leq \pi/2$, so $H(\eta_1, \eta_2) > 0$. It follows that $\hat{\phi}(\pi,\pi) \neq 0$. Consequently, the exact accuracy of ϕ is 4.

By using the methods in [12], we can easily prove that the subdivision scheme associated with mask a and dilation matrix M converges uniformly. Consequently, ϕ is a continuous function. We conclude that the optimal approximation order provided by $\mathbb{S}(\phi)$ is 4.

5. The subdivision and transition operators

We introduce two linear operators associated with a refinement equation. One is the subdivision operator, and the other is the transition operator. When the dilation matrix M is 2 times the identity matrix, the spectral properties of the subdivision and transition operators were studied in [10] and [18]. In this section, we extend the study to the case in which M is a general dilation matrix.

Let X and Y be two linear spaces, and T a linear mapping from X to Y. The **kernel** of T, denoted by $\ker(T)$, is the subspace of X consisting of all $x \in X$ such that $Tx = 0$.

Let a be an element in $\ell_0(\mathbb{Z}^s)$ and let M be a dilation matrix. The **subdivision operator** S_a is the linear operator on $\ell(\mathbb{Z}^s)$ defined by

$$S_a u(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)u(\beta), \qquad \alpha \in \mathbb{Z}^s,$$

where $u \in \ell(\mathbb{Z}^s)$. The **transition operator** T_a is the linear operator on $\ell_0(\mathbb{Z}^s)$ defined by

$$T_a v(\alpha) := \sum_{\beta\in\mathbb{Z}^s} a(M\alpha - \beta)v(\beta), \qquad \alpha \in \mathbb{Z}^s,$$

where $v \in \ell_0(\mathbb{Z}^s)$.

The following theorem shows that the subdivision operator S_a and the transition operator T_a have the same nonzero eigenvalues. We use I and I_0 to denote the identity mapping on $\ell(\mathbb{Z}^s)$ and $\ell_0(\mathbb{Z}^s)$, respectively.

Theorem 5.1. *The transition operator T_a has only finitely many nonzero eigenvalues. For $\sigma \in \mathbb{C} \setminus \{0\}$, the linear spaces $\ker(S_a - \sigma I)$ and $\ker(T_a - \sigma I_0)$ have the same dimension. In particular, σ is an eigenvalue of S_a if and only if it is an eigenvalue of T_a.*

Proof. For $N = 1, 2, \ldots$, let E_N denote the cube $[-N, N]^s$. Choose N such that E_{N-1} contains $\operatorname{supp} a := \{\alpha \in \mathbb{Z}^s : a(\alpha) \neq 0\}$. Let $K := \sum_{n=1}^\infty M^{-n}E_N$. In other words, x belongs to K if and only if $x = \sum_{n=1}^\infty M^{-n}y_n$ for some sequence of elements $y_n \in E_N$. Let $\ell(K)$ denote the linear space of all (finite) sequences on $K \cap \mathbb{Z}^s$. Consider the linear mapping A on $\ell(K)$ given by

$$Av(\alpha) := \sum_{\beta\in K\cap\mathbb{Z}^s} a(M\alpha - \beta)v(\beta), \qquad \alpha \in K \cap \mathbb{Z}^s,$$

where $v \in \ell(K)$. The dual mapping A' of A is given by

$$A'u(\beta) := \sum_{\alpha\in K\cap\mathbb{Z}^s} u(\alpha)a(M\alpha - \beta), \qquad \beta \in K \cap \mathbb{Z}^s,$$

where $u \in \ell(K)$. Let I_K denote the identity mapping on $\ell(K)$. Since $\ell(K)$ is finite dimensional, we have

$$\dim\big(\ker(A - \sigma I_K)\big) = \dim\big(\ker(A' - \sigma I_K)\big).$$

Thus, in order to establish the theorem, it suffices to prove the following two relations:

$$\dim\big(\ker(T_a - \sigma I_0)\big) = \dim\big(\ker(A - \sigma I_K)\big) \tag{5.1}$$

and

$$\dim\big(\ker(S_a - \sigma I)\big) = \dim\big(\ker(A' - \sigma I_K)\big). \tag{5.2}$$

For this purpose, we introduce the sets K_j $(j = 0, 1, \ldots)$ as follows:

$$K_j := M^{j-1}E_1 + \cdots + E_1 + K.$$

In particular, $K_0 = K$. Evidently, $K_j \subseteq K_{j+1}$ for $j = 0, 1, \ldots$, and $\mathbb{R}^s = \bigcup_{j=0}^\infty K_j$. Moreover,

$$M^{-1}(K_j + \operatorname{supp} a) \subseteq K_{j-1}, \qquad j = 1, 2, \ldots. \tag{5.3}$$

Indeed, we have $M^{-1}K + M^{-1}E_N = K$, and hence

$$M^{-1}(K_j + \operatorname{supp} a) \subseteq M^{j-2}E_1 + \cdots + E_1 + M^{-1}E_1 + M^{-1}K + M^{-1}E_{N-1} \subseteq K_{j-1}.$$

Suppose $\sigma \neq 0$ and $v \in \ker(T_a - \sigma I_0)$. Then $\operatorname{supp} v \subseteq K_j$ for some $j \geq 1$. We observe that $T_a v(\alpha) \neq 0$ implies $M\alpha - \beta \in \operatorname{supp} a$ for some $\beta \in K_j$. It follows that $\alpha \in M^{-1}(\operatorname{supp} a + K_j) \subseteq K_{j-1}$, by (5.3). In other words, $\operatorname{supp}(T_a v) \subseteq K_{j-1}$. Using this relation repeatedly, we obtain $\operatorname{supp}(T_a^j v) \subseteq K$. But $v = T_a v/\sigma = (T_a^j v)/\sigma^j$. Therefore, $\operatorname{supp} v \subseteq K$, and $v|_{K\cap\mathbb{Z}^s}$ belongs to $\ker(A - \sigma I_K)$. This shows that the restriction mapping $P : v \mapsto v|_{K\cap\mathbb{Z}^s}$ maps $\ker(T_a - \sigma I_0)$ to $\ker(A - \sigma I_K)$. Moreover, $v|_{K\cap\mathbb{Z}^s} = 0$ implies $v = 0$. So P is one-to-one. Let us show that P is also onto. Suppose $Aw = \sigma w$ for some $w \in \ell(K)$. Define $v(\alpha) := w(\alpha)$ for $\alpha \in K \cap \mathbb{Z}^s$ and $v(\alpha) := 0$ for $\alpha \in \mathbb{Z}^s \setminus K$. Then $T_a v = \sigma v$. Thus, P is one-to-one and onto, thereby establishing (5.1).

In order to prove (5.2), we consider the mapping $Q : u \mapsto u^*|_{K\cap\mathbb{Z}^s}$, where u^* is the sequence given by $u^*(\alpha) := u(-\alpha)$, $\alpha \in \mathbb{Z}^s$. Suppose $u \in \ker(S_a - \sigma I)$. Then

$$u(\alpha) = \frac{1}{\sigma}\sum_{\beta\in\mathbb{Z}^s} a(\alpha - M\beta)u(\beta), \qquad \alpha \in \mathbb{Z}^s.$$

It follows that

$$u^*(\alpha) = \frac{1}{\sigma}\sum_{\beta\in\mathbb{Z}^s} u^*(\beta)a(M\beta - \alpha), \qquad \alpha \in \mathbb{Z}^s.$$

For $\alpha \in K_j$ $(j \geq 1)$, $a(M\beta - \alpha) \neq 0$ only if $\beta \in M^{-1}(\operatorname{supp} a + K_j) \subseteq K_{j-1}$. Hence

$$u^*(\alpha) = \frac{1}{\sigma}\sum_{\beta\in K_{j-1}\cap\mathbb{Z}^s} u^*(\beta)a(M\beta - \alpha) \qquad \text{for } \alpha \in K_j \cap \mathbb{Z}^s. \tag{5.4}$$

This shows that $u^*|_{K\cap\mathbb{Z}^s}$ belongs to $\ker(A' - \sigma I_K)$. Thus, Q maps $\ker(S_a - \sigma I)$ to $\ker(A' - \sigma I_K)$. Moreover, if $u^*(\alpha) = 0$ for $\alpha \in K \cap \mathbb{Z}^s$, then it follows from (5.4) that $u^*(\alpha) = 0$ for $\alpha \in K_j \cap \mathbb{Z}^s$, $j = 1, 2, \ldots$. But $\mathbb{R}^s = \bigcup_{j=1}^\infty K_j$; hence $u^*(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s$. Thus, the mapping Q is one-to-one. It is also onto. Indeed, if $w \in \ker(A' - \sigma I_K)$, then

$$w(\alpha) = \frac{1}{\sigma}\sum_{\beta\in K\cap\mathbb{Z}^s} w(\beta)a(M\beta - \alpha), \qquad \alpha \in K \cap \mathbb{Z}^s.$$

For $\alpha \in K \cap \mathbb{Z}^s$, let $u^*(\alpha) := w(\alpha)$; for $\alpha \in (K_j \setminus K_{j-1}) \cap \mathbb{Z}^s$ $(j = 1, 2, \ldots)$, let $u^*(\alpha)$ be determined recursively by (5.4). Then $u \in \ker(S_a - \sigma I)$ and $Qu = w$. Thus, Q is one-to-one and onto, so that (5.2) is valid. The proof of the theorem is complete. □

A sequence u on $\mathbb{Z}^s$ is called a **polynomial sequence** if there exists a polynomial p such that $u(\alpha) = p(\alpha)$ for all $\alpha \in \mathbb{Z}^s$. The degree of u is the same as the degree of p. For a nonnegative integer k, let P_k be the linear space of all polynomial sequences of degree at most k, and let

$$V_k := \Big\{ v \in \ell_0(\mathbb{Z}^s) : \sum_{\alpha\in\mathbb{Z}^s} p(\alpha)v(\alpha) = 0 \;\forall\, p \in \Pi_k \Big\}.$$

For $u \in \ell(\mathbb{Z}^s)$ and $v \in \ell_0(\mathbb{Z}^s)$, we define

$$\langle u, v\rangle := \sum_{\alpha\in\mathbb{Z}^s} u(\alpha)v(\alpha).$$

Theorem 5.2. *Let M be an $s \times s$ dilation matrix and Ω a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M^T\mathbb{Z}^s$. For any $a \in \ell_0(\mathbb{Z}^s)$, the following statements are equivalent:*
(a) The sequence a satisfies the sum rules of order $k+1$.
(b) V_k is invariant under the transition operator T_a.
(c) P_k is invariant under the subdivision operator S_a.
(d) $D^\mu H(2\pi(M^T)^{-1}\omega) = 0$ for all $|\mu| \le k$ and all $\omega \in \Omega \setminus \{0\}$.

Proof. (a) $\Rightarrow$ (b): Let $p \in \Pi_k$ and $v \in V_k$. We have

$$\sum_{\alpha\in\mathbb{Z}^s} p(\alpha)T_a v(\alpha) = \sum_{\beta\in\mathbb{Z}^s}\Big[\sum_{\alpha\in\mathbb{Z}^s} p(\alpha)a(M\alpha-\beta)\Big]v(\beta).$$

Let $q(x) := p(M^{-1}x)$, $x \in \mathbb{R}^s$. Then $p(x) = q(Mx)$, $x \in \mathbb{R}^s$. By Taylor's formula, we have

$$q(M\alpha) = q(M\alpha - \beta + \beta) = \sum_{|\mu|\le k} q_\mu(M\alpha-\beta)\beta^\mu,$$

where $q_\mu := D^\mu q/\mu! \in \Pi_k$. Hence

$$\sum_{\alpha\in\mathbb{Z}^s} p(\alpha)a(M\alpha-\beta) = \sum_{\alpha\in\mathbb{Z}^s} q(M\alpha)a(M\alpha-\beta) = \sum_{|\mu|\le k} c_\mu\beta^\mu,$$

where

$$c_\mu := \sum_{\alpha\in\mathbb{Z}^s} q_\mu(M\alpha-\beta)a(M\alpha-\beta)$$

is independent of β, by condition (a). Thus, we obtain

$$\sum_{\alpha\in\mathbb{Z}^s} p(\alpha)T_a v(\alpha) = \sum_{|\mu|\le k} c_\mu \sum_{\beta\in\mathbb{Z}^s}\beta^\mu v(\beta) = 0,$$

because $v \in V_k$. This shows that $T_a v \in V_k$ for $v \in V_k$. In other words, V_k is invariant under T_a.

(b) $\Rightarrow$ (c): Suppose $p \in P_k$. We wish to show that $u := S_a p$ lies in P_k. We claim that $\langle u, v\rangle = 0$ for all $v \in V_k$. Indeed,

$$\begin{aligned}\langle u, v\rangle &= \sum_{\alpha\in\mathbb{Z}^s} u(\alpha)v(\alpha) = \sum_{\alpha\in\mathbb{Z}^s}\sum_{\beta\in\mathbb{Z}^s} a(\alpha-M\beta)p(\beta)v(\alpha)\\ &= \sum_{\beta\in\mathbb{Z}^s} p(-\beta)\sum_{\alpha\in\mathbb{Z}^s} a(M\beta-\alpha)v(-\alpha) = \sum_{\beta\in\mathbb{Z}^s} p(-\beta)w(\beta),\end{aligned}$$

where $w := T_a v^*$ with v^* given by $v^*(\alpha) = v(-\alpha)$, $\alpha \in \mathbb{Z}^s$. Since V_k is invariant under T_a and $v^* \in V_k$, we have $w \in V_k$. It follows that

$$\langle u, v\rangle = \sum_{\beta\in\mathbb{Z}^s} p(-\beta)w(\beta) = 0.$$

For a multi-index μ with $|\mu| = k+1$, we have $\nabla^\mu\delta_\alpha \in V_k$ for all $\alpha \in \mathbb{Z}^s$. Hence $\langle u, \nabla^\mu\delta_\alpha\rangle = 0$. In other words, $\nabla^\mu u(\alpha) = 0$ for all $\alpha \in \mathbb{Z}^s$ and $|\mu| = k+1$. This shows that u is a polynomial sequence of degree at most k.

(c) $\Rightarrow$ (a): For $p \in \Pi_k$, let $q(\gamma) := \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)\, p(M\beta+\gamma)$ for $\gamma \in \mathbb{Z}^s$. We claim that q is a polynomial sequence. Indeed, by using Taylor's formula, we have

$$p(M\beta+\gamma) = \sum_{|\mu|\le k} t_\mu(M\beta)\gamma^\mu,$$

where $t_\mu := D^\mu p/\mu!$. Set $q_\mu(\beta) := t_\mu(-M\beta)$ for $\beta \in \mathbb{Z}^s$. Then for $\gamma \in \mathbb{Z}^s$,

$$\begin{aligned} q(\gamma) &= \sum_{\beta\in\mathbb{Z}^s} a(M\beta+\gamma)\, p(M\beta+\gamma) \\ &= \sum_{\beta\in\mathbb{Z}^s} \sum_{|\mu|\le k} a(\gamma+M\beta)\, q_\mu(-\beta)\gamma^\mu = \sum_{|\mu|\le k} (S_a q_\mu)(\gamma)\, \gamma^\mu. \end{aligned}$$

Note that q_μ is a polynomial sequence of degree at most k. By condition (c), $S_a q_\mu$ is a polynomial sequence; hence so is q. We observe that $q(\gamma+M\eta) = q(\gamma)$ for all $\eta \in \mathbb{Z}^s$ and $\gamma \in \mathbb{Z}^s$, that is, q is a constant sequence on the lattice $\gamma + M\mathbb{Z}^s$ for each $\gamma \in \mathbb{Z}^s$. Hence q itself must be a constant sequence. This verifies condition (a).

Finally, the equivalence between (a) and (d) was proved in Lemma 3.3. $\square$

We remark that the equivalence between (c) and (d) was proved in [7, p. 98] for the case when the dilation matrix M is 2 times the identity matrix.

6. Smoothness and approximation order

In this section we discuss the relationship between approximation and smoothness properties of a refinable function.

Suppose ϕ satisfies the refinement equation (1.1) with the dilation matrix M being 2 times the identity matrix. It was proved by Jia in [18] that $\phi \in W_1^k(\mathbb{R}^s)$ and $\hat\phi(0) \ne 0$ imply that $\Pi_k \subset \mathbb{S}(\phi)$ and $\mathbb{S}(\phi)$ provides approximation order $k+1$. This result improves an earlier result of Cavaretta, Dahmen, and Micchelli about polynomial reproducibility of smooth refinable functions (see [7, p. 158]).

The above results can be extended to the case in which the dilation matrix is isotropic. Let M be an $s\times s$ matrix with its entries in $\mathbb{C}$. We say that M is **isotropic** if M is similar to a diagonal matrix diag $\{\lambda_1, \dots, \lambda_s\}$ with $|\lambda_1| = \cdots = |\lambda_s|$. For example, for $a, b \in \mathbb{R}$, the matrix

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

is isotropic. Obviously, a matrix M is isotropic if and only if its transpose M^T is isotropic.

Lemma 6.1. *Let M be an isotropic matrix with spectral radius σ. For any vector norm $\|\cdot\|$ on $\mathbb{R}^s$, there exist two positive constants C_1 and C_2 such that the inequalities*

$$C_1\sigma^n\|v\| \le \|M^n v\| \le C_2\sigma^n\|v\|$$

hold true for every positive integer n and every vector $v \in \mathbb{R}^s$.

Proof. Since M is isotropic, we can find a basis $\{v_1, \dots, v_s\}$ for $\mathbb{C}^s$ such that $Mv_j = \lambda_j v_j$ with $|\lambda_1| = \cdots = |\lambda_s| = \sigma$. Recall that two norms on a finite-dimensional linear space are equivalent. Hence there exist two positive constants C_1 and C_2 such that

$$C_1 \sum_{j=1}^s |a_j| \le \|v\| \le C_2 \sum_{j=1}^s |a_j| \qquad \text{for } v = \sum_{j=1}^s a_j v_j.$$

But for $v = \sum_{j=1}^s a_j v_j$ we have $M^n v = \sum_{j=1}^s a_j \lambda_j^n v_j$. It follows that

$$\|M^n v\| \le C_2 \sum_{j=1}^s |a_j \lambda_j^n| = C_2 \sigma^n \sum_{j=1}^s |a_j| \le C_2 C_1^{-1} \sigma^n \|v\|$$

and

$$\|M^n v\| \ge C_1 \sum_{j=1}^s |a_j \lambda_j^n| = C_1 \sigma^n \sum_{j=1}^s |a_j| \ge C_1 C_2^{-1} \sigma^n \|v\|.$$

This completes the proof of the lemma.

Lemma 6.2. *Let M be an isotropic matrix with spectral radius σ. For an infinitely differentiable function f on $\mathbb{R}^s$, let*

$$f_n(\xi) := f\big((M^T)^n \xi\big), \qquad \xi \in \mathbb{R}^s, \quad n = 0, 1, 2, \dots .$$

Then, for each positive integer r, there exists a positive constant C depending only on r and the matrix M such that

$$\max_{|\mu|=r} \big|D^\mu f_n(\xi)\big| \le C\, \sigma^{rn} \max_{|\nu|=r} \big|D^\nu f\big((M^T)^n \xi\big)\big| \qquad \forall\, \xi \in \mathbb{R}^s. \tag{6.1}$$

Proof. Let $B = (b_{pq})_{1 \le p,q \le s}$ be the matrix $(M^T)^n$. By the chain rule, for $j = 1, \dots, s$, we have

$$D_j f_n(\xi) = (b_{1j} D_1 + \cdots + b_{sj} D_s) f\big((M^T)^n \xi\big), \qquad \xi \in \mathbb{R}^s.$$

Hence, for a multi-index $\mu = (\mu_1, \dots, \mu_s)$ with $|\mu| = r$,

$$D^\mu f_n(\xi) = \prod_{j=1}^s D_j^{\mu_j} f_n(\xi) = \prod_{j=1}^s (b_{1j} D_1 + \cdots + b_{sj} D_s)^{\mu_j} f\big((M^T)^n \xi\big), \qquad \xi \in \mathbb{R}^s.$$

By Lemma 6.1, there exists a constant $C_1 > 0$ depending only on the matrix M such that $|b_{pq}| \le C_1 \sigma^n$ for all p, q. We may express $\prod_{j=1}^s (b_{1j} D_1 + \cdots + b_{sj} D_s)^{\mu_j}$ as $\sum_{|\nu|=r} c_\nu D^\nu$, where each c_ν is a linear combination of products of r factors of the b_{pq}'s. Hence there exists a positive constant C depending only on r and the matrix M such that $|c_\nu| \le C \sigma^{rn}$ for all $|\nu| = r$. This proves (6.1). □

Now we are in a position to establish the main result of this section.

Theorem 6.3. *Suppose M is an $s \times s$ isotropic dilation matrix, and a is an element in $\ell_0(\mathbb{Z}^s)$ satisfying (1.2). Let ϕ be the normalized solution of the refinement equation (1.1). If $\phi \in W_1^k(\mathbb{R}^s)$, then $\Pi_k \subset \mathbb{S}(\phi)$ and $\mathbb{S}(\phi)$ provides approximation order $k+1$.*

Proof. Since $\hat{\phi}(0) = 1$, in order to prove $\mathbb{S}(\phi) \supset \Pi_k$, it suffices to show that for $|\mu| \le k$,

$$D^\mu \hat{\phi}(2\pi\beta) = 0 \qquad \forall\, \beta \in \mathbb{Z}^s \setminus \{0\}. \tag{6.2}$$

The proof proceeds with induction on $|\mu|$, the length of μ.

Let H be the function given in (3.2). A repeated application of (3.1) yields that, for $n = 1, 2, \dots$,

$$\hat{\phi}(\xi) = \left[\prod_{j=1}^n H\big((M^T)^{-j} \xi\big)\right] \hat{\phi}\big((M^T)^{-n} \xi\big), \qquad \xi \in \mathbb{R}^s.$$

It follows that

$$\hat{\phi}\big((M^T)^n\xi\big) = h_n(\xi)\hat{\phi}(\xi), \qquad \xi \in \mathbb{R}^s, \tag{6.3}$$

where $h_n(\xi) := \prod_{j=1}^n H\big((M^T)^{j-1}\xi\big)$. Note that H is 2π-periodic and $H(0) = 1$. Thus, we have

$$\hat{\phi}\big(2\pi(M^T)^n\beta\big) = \left[\prod_{j=1}^n H\big(2\pi(M^T)^{j-1}\beta\big)\right]\hat{\phi}(2\beta\pi) = \hat{\phi}(2\beta\pi), \qquad \beta \in \mathbb{Z}^s.$$

If $\phi \in L_1(\mathbb{R}^s)$, then by the Riemann-Lebesgue lemma we obtain

$$\hat{\phi}(2\beta\pi) = \lim_{n\to\infty} \hat{\phi}\big(2\pi(M^T)^n\beta\big) = 0 \qquad \forall\, \beta \in \mathbb{Z}^s \setminus \{0\}.$$

This establishes (6.2) for $\mu = 0$.

Let $0 < r \le k$. Assume that (6.2) has been proved for $|\mu| < r$. We wish to establish (6.2) for $|\mu| = r$. For this purpose, we deduce from (6.3) that

$$\hat{\phi}(\xi) = f_n(\xi)\,\big[1/h_n(\xi)\big], \qquad \xi \in \mathbb{R}^s,$$

where $f_n(\xi) := \hat{\phi}((M^T)^n\xi)$, $\xi \in \mathbb{R}^s$. By using the Leibniz formula for differentiation, we get

$$D^\mu\hat{\phi}(\xi) = \sum_{\nu\le\mu}\binom{\mu}{\nu} D^\nu f_n(\xi)\, D^{\mu-\nu}[1/h_n](\xi), \qquad \xi \in \mathbb{R}^s. \tag{6.4}$$

But, for $\beta \in \mathbb{Z}^s \setminus \{0\}$ and $|\nu| < r$, we have $D^\nu f_n(2\pi\beta) = 0$, by the induction hypothesis. When $\nu = \mu$, we have $[1/h_n](2\pi\beta) = 1$. Hence it follows from (6.4) that

$$D^\mu\hat{\phi}(2\pi\beta) = D^\mu f_n(2\pi\beta), \qquad \beta \in \mathbb{Z}^s \setminus \{0\}. \tag{6.5}$$

By Lemma 6.2, we have

$$\big|D^\mu f_n(2\pi\beta)\big| \le C\,\sigma^{rn} \max_{|\nu|=r}\big|D^\nu\hat{\phi}\big((M^T)^n 2\pi\beta\big)\big|, \qquad \beta \in \mathbb{Z}^s \setminus \{0\}, \tag{6.6}$$

where $C > 0$ is a constant independent of n.

In what follows, we use v_j to denote the jth coordinate of a vector v in $\mathbb{R}^s$. For a multi-index $\nu = (\nu_1, \dots, \nu_s)$, let ϕ_ν be the function given by $\phi_\nu(x) = (-ix)^\nu\phi(x)$, $x \in \mathbb{R}^s$. Then $D^\nu\hat{\phi} = \hat{\phi}_\nu$ and

$$\big((-iD_j)^r\phi_\nu\big)\hat{}\,(\xi) = \xi_j^r D^\nu\hat{\phi}(\xi), \qquad \xi = (\xi_1, \dots, \xi_s) \in \mathbb{R}^s.$$

Since $\phi \in W_1^k(\mathbb{R}^s)$, we have $(-iD_j)^r\phi_\nu \in L_1(\mathbb{R}^s)$. Thus, by the Riemann-Lebesgue lemma, we obtain

$$\lim_{n\to\infty}\big((M^T)^n\beta\big)_j^r\, D^\nu\hat{\phi}\big(2\pi(M^T)^n\beta\big) = 0 \qquad \text{for } \beta \in \mathbb{Z}^s \setminus \{0\}.$$

This is true for $j = 1, \dots, s$; hence it follows that

$$\lim_{n\to\infty} \|(M^T)^n\beta\|^r D^\nu\hat{\phi}\big(2\pi(M^T)^n\beta\big) = 0 \qquad \text{for } \beta \in \mathbb{Z}^s \setminus \{0\},$$

where $\|\cdot\|$ is a vector norm on $\mathbb{R}^s$. By Lemma 6.1, there exists a positive constant $C_1 > 0$ independent of n such that

$$C_1\sigma^n\|\beta\| \le \|(M^T)^n\beta\|.$$

Therefore

$$\lim_{n\to\infty} \sigma^{nr} D^{\nu}\hat{\phi}\big(2\pi(M^T)^n\beta\big) = 0 \qquad \text{for } \beta \in \mathbb{Z}^s \setminus \{0\}.$$

This in connection with (6.5) and (6.6) tells us that $D^{\mu}\hat{\phi}(2\pi\beta) = 0$ for $|\mu| = r$ and $\beta \in \mathbb{Z}^s \setminus \{0\}$. The proof of the theorem is complete. □

Recall that Ω is a complete set of representatives of the distinct cosets of $\mathbb{Z}^s/M^T\mathbb{Z}^s$. Thus, as a consequence of Theorem 6.3, we conclude that if the normalized solution ϕ of the refinement equation (1.1) lies in $W_1^k(\mathbb{R}^s)$, and if $N(\phi) \cap (2\pi(M^T)^{-1}\Omega) = \emptyset$, then the refinement mask a satisfies all the conditions in Theorem 5.2.

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